REFLEXIVITY OF OPERATOR ALGEBRAS OF FINITE SPLIT STRICT MULTIPLICITY

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Abstract. We study the invariant subspaces of abelian operator algebras of finite split strict multiplicity. We give sufficient conditions for the reflexivity and hereditary reflexivity of these algebras.

Let $A \subset B(X)$ be a strongly closed algebra of operators on a Banach space X. We denote by $Lat(A)$ the set of all closed subspaces of X that are invariant for every $a \in A$. Let $algLat(A) = \{T \in B(X) \mid TK \subseteq K$, for every $K \in Lat(A)\}$.

An algebra A is called reflexive if $A = algLat(A)$. More generally, a subspace $\mathcal{L} \subseteq B(X)$ is called reflexive if for every $T \in B(X)$ such that $Tx \in \overline{\mathcal{L}x}$, for every $x \in X$, it follows that $T \in \mathcal{L}$. An algebra A is called hereditarily reflexive if every strongly closed subspace $\mathcal{L} \subseteq A$ is reflexive.

In [4], A. Lambert initiated the study of strictly cyclic algebras and proved that a unital, strictly cyclic, semisimple operator algebra on a Hilbert space is reflexive. In [6] the result of Lambert is extended to the case of Banach spaces and is proven that every norm closed, abelian, unital, semisimple, strictly cyclic algebra is in addition hereditarily reflexive. In this paper we investigate the similar problem for algebras of finite strict multiplicity. In [3] an algebra $A \subset B(X)$ is said to be of strict multiplicity $p \in \mathbb{N}$ (denoted by $s.m.(A) = p$), if there are $x_1, x_2, ..., x_p \in X$ such that $X =$ linear span $\{ax_i \mid a \in A;\ i = 1, 2, ..., p\}$.

We will consider a special case of algebras of finite strict multiplicity, namely algebras of finite split strict multiplicity. An algebra $A \subset B(X)$ is said to be of split strict multiplicity $p \in \mathbb{N}$ (denoted by $s.s.m(A) = p$) if there are $x_1, x_2, ..., x_p \in X$ such that the subspaces $\{ax_i \mid a \in A\} = X_i$ are closed and $X = \sum_{i=1}^{p} \oplus X_i$.

Proposition 1 and Remark 2 below provide a characterization of abelian operator algebras of split strict multiplicity 2. In particular, if $A \subseteq B(X)$ is strictly cyclic, then $A^{(2)} = \{a \oplus a \mid a \in A\} \subseteq B(X \oplus X)$ is the simplest example of an algebra of split strict multiplicity 2.

Proposition 1. *Let $A \subseteq B(X)$ be an abelian, unital, norm closed algebra of strict split multiplicity p. Then there exist closed ideals $I_k \subseteq A$, $k = 1, 2, ..., p$ and Banach space isomorphisms $S_k : A/I_k \to Ax_k$ such that:*

(i) $S_k(\varphi_k(a)) = ax_k$.

(ii) $S = \sum_{k=1}^{p} \oplus S_k : \sum_{k=1}^{p} \oplus A/I_k \to X$ *is a Banach space isomorphism.*

2000 *Mathematics Subject Classification.* Primary 47A15; Secondary 47L75.

(iii) *The mapping* $T : A \to B(\sum_{k=1}^{p} \oplus A/I_k)$ *defined by* $Ta = S^{-1}aS$ *is a Banach algebra isomorphism.*

(iv) *If* $p = 2$, $I_1 + I_2$ *is closed.*

Proof. Let $I_k = \{a \in A \mid ax_k = 0\}$, $k = 1, 2, ..., p$. Clearly, I_k are closed ideals of A. Let $S_k : A/I_k \to Ax_k$ be defined by $S_k(\varphi_k(a)) = ax_k$. Then S_k is well defined, one to one and onto for all $k = 1, 2, ..., p$. To see that S_k is continuous, let $\{a_n\}_n \subset A$ be such that $\varphi_k(a_n) \underset{n}{\to} 0$. Then there is a sequence $\{i_{k,n}\}_n \subset I_k$ such that $a_n + i_{k,n} \underset{n}{\to} 0$ in the norm of A. It the follows that $(a_n + i_{k,n})x_k \underset{n}{\to} 0$. Since $i_{k,n} \in I_k$ it follows that $a_nx_k \underset{n}{\to} 0$, so S_k is a continuous, one-to-one, onto mapping between the Banach spaces A/I_k and Ax_k. By the open mapping theorem, S_k is a Banach space isomorphism.

Conditions (ii) and (iii) are straightforward to ckeck. We verify next condition (iv).

Let $a_n = i_{1,n} + i_{2,n} \underset{n}{\to} a_0$, where $\{i_{1,n}\} \subset I_1$ and $\{i_{2,n}\} \subset I_2$. Then, in particular, the restriction $a_n|_{X_1} = i_{2,n}|_{X_1} \underset{n}{\to} a_0|_{X_1}$. On the other hand, $i_{2,n}|_{X_2} = 0$ for every $n \in \mathbb{N}$. It then follows that the sequence $\{i_{2,n}\}$ is convergent in norm. Since I_2 is norm closed, it follows that there exists $i_2 \in I_2$ such that $i_{2,n} \underset{n}{\to} i_2$. Hence $a_n - i_{2,n} \to a_0 - i_2$ in norm and since $a_n - i_{2,n} = i_{1,n} \in I_1$ and I_1 is closed, we have $a_0 - i_2 = i_1 \in I_1$, and thus $a_0 = i_1 + i_2$ and $I_1 + I_2$ is closed as claimed.

□

Remark 2. Let A be an abelian, unital Banach algebra and I_1, I_2 be two closed ideals such that $I_1 + I_2$ is closed and $I_1 \cap I_2 = (0)$. Then the mapping $a \to T_a \in B(A/I_1 \oplus A/I_2)$ where $T_a(\varphi_1(a_1) \oplus \varphi_2(a_2)) = \varphi_1(aa_1) \oplus \varphi_2(aa_2)$ is a continuous embedding of A into $B(A/I_1 \oplus A/I_2)$.

By Proposition 1, any abelian, unital, norm closed $A \subseteq B(X)$, (X Banach space) with $s.s.m(A) = 2$ is spatially isomorphic with $\{T_a \mid a \in A\}$ if $I_k = \{a \in A \mid ax_k = 0\}$, for $k = 1, 2$.

Remark 3. (i) Let $A \subseteq B(X)$ be an abelian, unital, norm closed subalgebra with $s.s.m.(A) = 2$. Then, the subspace $D_2 = \{ax_1 \oplus ax_2 \mid a \in A\} \subseteq X$ is closed.

Indeed, let $\{a_n\} \subset A$ be such that $a_nx_1 \oplus a_nx_2 \to ax_1 \oplus bx_2 \in X$. Then, by Proposition 1, $\varphi_1(a_n \to \varphi_1(a)$ and $\varphi_2(a_n) \to \varphi_2(b)$. Therefore, there are sequences $\{i_{1,n}\} \subset I_1$, $\{i_{2,n}\} \subset I_2$ such that $a_n + i_{1,n} \underset{n}{\to} a$ and $a_n + i_{2,n} \underset{n}{\to} b$. Hence $i_{1,n} - i_{2,n} \underset{n}{\to} a - b$. By Proposition 1 (iv), $a - b \in I_1 + I_2$, so there exist $i_1 \in I_1$, $i_2 \in I_2$ such that $a - b = i_1 - i_2$. Hence $a - i_1 = b - i_2$. Let $a_0 = a - i_1 = b - i_2$. Then $a_0x_1 = ax_1$ and $a_0x_2 = bx_2$, so $a_nx_1 \oplus a_nx_2 \underset{n}{\to} a_0x_1 \oplus a_0x_2 \in D_2$.

(ii) Let $A \subseteq B(X)$ be an abelian, unital, strongly closed subalgebra with $s.s.m.(A) = p \in \mathbb{N}$. Then $D_p = \{ax_1 \oplus ax_2 \oplus ... \oplus ax_p \mid a \in A\} \subseteq X$ is closed.

Indeed, let $\{a_n\} \subset A$ be such that $a_nx_1 \oplus a_nx_2 \oplus ... \oplus a_nx_p \underset{n}{\to} b_1x_1 \oplus ... \oplus b_px_p \in X$. Then, since A is abelian, $a_nc_jx_j \to b_jc_jx_j$ for every $c_j \in A$, $j = 1, 2, ..., p$. Hence $a_n(\sum_{j=1}^{p} c_jx_j) \to \sum_{j=1}^{p} b_jc_jx_j$. Therefore the sequence $\{a_nx\}$ is convergent for every $x \in X$. Since A is strongly closed, there is $a_0 \in A$ such that $a_nx \to a_0x$ for all $x \in X$. Hence $a_nx_1 \oplus a_nx_2 \oplus ... \oplus a_nx_p = a_n(x_1 \oplus ... \oplus x_p) \to a_0(x_1 \oplus ... \oplus x_p) = a_0x_1 \oplus a_0x_2 \oplus ... \oplus a_0x_p \in D_p$

(iii) Let $A \subseteq B(X)$ be an abelian, unital, norm closed subalgebra with $s.s.m.(A) = p \in \mathbb{N}$. Let $\mathcal{M} \subseteq A$ be a maximal ideal. Then the subspaces $\mathcal{M}x_j = \{mx_j \mid m \in \mathcal{M}\} \subseteq X$, $j = 1, ..., p$, are closed.

Indeed, the statement follows if we show that $\mathcal{M} + I_j$ is closed. If $I_j \subseteq \mathcal{M}$ then $\mathcal{M} + I_j = \mathcal{M}$. If $I_j \not\subseteq \mathcal{M}$ then $\mathcal{M} + I_j = A$. In either case $\mathcal{M} + I_j$ is closed.

In what follows we will denote by $\mathcal{R}_j$ the radical of the (abelian, unital) Banach algebra A/I_j, i.e. the intersection of all the maximal ideals of A/I_j. The next result shows that every abelian, unital, strongly closed subalgebra $A \subseteq B(X)$, with $s.s.m.(A) = p \in \mathbb{N}$ is reflexive modulo the radicals.

By Proposition 1 we can assume that $X = \sum_{j=1}^{p} \oplus A/I_j$ and $T_a(\sum_{j=1}^{p} \oplus \varphi_j(a_j)) = \sum_{j=1}^{p} \oplus \varphi_j(aa_j)$ for $a \in A$.

Theorem 4. *Let $A \subseteq B(X)$ be an abelian, unital, strongly closed subalgebra with $s.s.m.(A) = p \in \mathbb{N}$. For every $T \in algLat(A)$ there is $a_0 \in A$ such that $(T - T_{a_0})(x) \in \sum_{j=1}^{p} \oplus \mathcal{R}_j$ for every $x \in X$.*

Proof. With the identification $X = \sum_{j=1}^{p} \oplus A/I_j$ and $A = \{T_a \mid a \in A\}$ from Proposition 1, let $T \in algLat(A)$. By Remark 3 (ii) , $D_p \in Lat(A)$. Since $\sum_{j=1}^{p} \oplus \varphi_j(1) \in D_p$, there is $a_0 \in A$ such that $T(\sum_{j=1}^{p} \oplus \varphi_j(1)) = \sum_{j=1}^{p} \oplus \varphi_j(a_0)$. In particular, if $j \in \{1, 2, ..., p\}$, $T(\varphi_j(1)) = \varphi_j(a_0)$. Let $\mathcal{M} \subseteq A$ be a maximal ideal and Ψ the corresponding multiplicative functional on A with $ker\Psi = \mathcal{M}$.

Let $a \in A$. Then $x = \Psi(a)1 - a \in \mathcal{M}$ since $\Psi(x) = 0$. By Remark 3 (iii), $\varphi_j(\mathcal{M}) \in Lat(A)$. Therefore $\varphi_j(x) = \Psi(a)\varphi_j(1) - \varphi_j(a) \in \varphi_j(\mathcal{M})$ and $T(\varphi_j(x)) = \Psi(a)T(\varphi_j(1)) - T(\varphi_j(a)) \in \varphi_j(\mathcal{M})$. Hence

$$\Psi(a)\varphi_j(a_0) - T(\varphi_j(a)) \in \varphi_j(\mathcal{M}) \tag{1}$$

Since $T \in algLat(A)$, $T(\varphi_j(A)) \subseteq \varphi_j(A)$. Let $b \in A$ be such that $T(\varphi_j(a)) = \varphi_j(b)$. Using relation 1 above we get

$$\Psi(a)a_0 - b \in \mathcal{M} + I_j \text{ or} \tag{2}$$

$$\Psi(a)a_0 - b + i \in \mathcal{M} \text{ for some } i \in I_j. \tag{3}$$

Applying Ψ to relation 3 we get

$$\Psi(aa_0 - b + i) = 0. \tag{4}$$

Hence $aa_0 - b \in \mathcal{M} + I_j$. It follows that $\varphi_j(aa_0) - T(\varphi_j(a)) \in \varphi_j(\mathcal{M})$. Since this holds in particular for every maximal ideal $\mathcal{M} \supseteq I_j$, it follows that $T(\varphi_j(a)) - T_{a_0}(\varphi_j(a)) \in \mathcal{R}_j$ and the proof is complete.

□

Notice that in the above theorem we assumed that the algebra $A \subseteq B(X)$ is strongly closed. The next result shows that if $s.s.m.(A) = 2$ we can assume that A ia only norm closed.

Corollary 5. *Let $A \subseteq B(X)$ be an abelian, unital, norm closed subalgebra with $s.s.m.(A) = 2$. For every $T \in algLat(A)$ there is $a_0 \in A$ such that $(T - T_{a_0})(x) \in \mathcal{R}_1 \oplus \mathcal{R}_2$ for every $x \in X$.*

Proof. The only modification in the proof of Theorem 4 is that if A is norm closed, Remark 3 (i) (instead of (ii)) can be applied. □

If A has spectral synthesis (see [1, 5] for examples of abelian algebras that have spectral synthesis) then $\mathcal{R}_j = (0)$ for all $j \in \{1, 2, ..., p\}$ and we have the following:

Corollary 6. *Let $A \subseteq B(X)$ be an abelian, unital, strongly closed subalgebra with spectral synthesis. If $s.s.m.(A) = p \in \mathbb{N}$, then A is reflexive.*

Corollary 7. *Let $A \subseteq B(X)$ be an abelian, unital, norm closed subalgebra with spectral synthesis. If $s.s.m.(A) = 2$, then A is reflexive.*

We will study next the hereditary reflexivity of abelian algebras of finite split strict multiplicity $A \subseteq B(X)$. We will assume that A is semisimple, i.e. that the intersection of all maximal, modular ideals is (0).

It is known from [7] that any semisimple, abelian, Banach algebra has a unique norm topology. Notice that if A is semisimple, the quotients of A may not be semisimple (see [2]).

Theorem 8. *Let $A \subseteq B(X)$ be an abelian, unital, semisimple, strongly closed subalgebra with $s.s.m.(A) = p \in \mathbb{N}$. Let $L \subseteq A$ be a norm closed subspace. If $T \in B(X)$ is such that $Tx \in \overline{Lx}$ for every $x \in X$, then there exists $l_0 \in L$ such that $(T - T_{l_0})(x) \in \sum_{j=1}^{p} \oplus \mathcal{R}_j$ for every $x \in X$.*

Proof. By Remark 3 (ii) D_p is closed. By Proposition 1 we can assume that $X = \sum_{j=1}^{p} \oplus A/I_j$ and $D_p = \{\sum_{j=1}^{p} \oplus \varphi_j(a) |\ a \in A\}$.

For $a \in A$ set

$$|||a||| = \sum_{j=1}^{p} \|\varphi_j(a)\|.$$

It is easy to see that $||| \cdot |||$ is an algebra norm on A. Since D_p is closed, $||| \cdot |||$ is a Banach algebra norm. By ([7] Corollary 2.5.18) A has a unique norm topology. Hence $||| \cdot |||$ is equivalent to the original norm of $A \subseteq B(X)$.

Let now $x_0 = \sum_{j=1}^{p} \oplus \varphi_j(1)$. Then, in particular, $Tx_0 \in \overline{Lx_0}$. There exists a sequence $\{l_n\} \subset L$ such that $Tx_0 = \lim_n \sum_{j=1}^{p} \oplus \varphi_j(l_n)$.

Since D_p is closed, $l_n \xrightarrow{|||\cdot|||} l_0 \in A$. Therefore $l_n \to l_0 \in A$ in the original norm of A. Since L is closed, $l_0 \in L$. Hence

$$Tx_0 = T(\sum_{j=1}^{p} \oplus \varphi_j(1)) = \sum_{j=1}^{p} \oplus \varphi_j(l_0).$$

Repeating now the proof of Theorem 4, we get $(T - T_{l_0})(x) \in \sum_{j=1}^{p} \oplus \mathcal{R}_j$ for every $x \in X$.

□

Using now Remark 3 (i) instead of (ii), we get:

Corollary 9. *Let $A \subseteq B(X)$ be an abelian, unital, semisimple, norm closed subalgebra with s.s.m.$(A) = 2$. Let L be a norm closed subspace. If $T \in B(X)$ is such that $Tx \in \overline{Lx}$ for every $x \in X$, then there exists $l_0 \in L$ such that $(T - T_{l_0})(x) \in \mathcal{R}_1 \oplus \mathcal{R}_2$ for every $x \in X$.*

If $\mathcal{R}_j = (0)$ for all $j = 1, 2, ..., p$, in particular if A has spectral synthesis, we get:

Corollary 10. *Let $A \subseteq B(X)$ be an abelian, unital, strongly closed subalgebra with spectral synthesis. If s.s.m.$(A) = p \in \mathbb{N}$, then A is hereditarily reflexive.*

Corollary 11. *Let $A \subseteq B(X)$ be an abelian, unital, norm closed subalgebra with spectral synthesis. If s.s.m.$(A) = 2$, then A is hereditarily reflexive.*

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